Math 126 Midterm 1 Problems

P.1) True/False

A.) The method of characteristics guarantees a solution to

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(0,x) = \begin{cases} 1 & x \leq 0 \\ 1-x & x \in (0,1) \\ 0 & x \geq 1 \end{cases} \end{cases}$$

for all time.

$\boxed{\text{False - HW 1 Problem 4}}$

B.) If you lived in 2-Dimensions and someone clapped, you would hear a sharp noise followed by silence.

False - Interpreting Huygen's Priciple - See lecture 8 or figure 4.14 (page 66) in the text

C.) Let $U \subset \mathbb{R}^n$ be a bounded domain with piecewise-C^1 boundary. A solution $u \in C^2([0,\infty)\times\overline{U})$ to

$$\begin{cases} \partial^2 u/\partial t^2 - c^2 \Delta u = f \\ u(0,x) = g \\ \partial_t u(0,x) = h \\ u|_{\partial U} = 0 \end{cases}$$

is unique for $f \in C^0$, $g \in C^3$, $h \in C^1$.

$\boxed{\text{True - Corollary 4.13}}$

D.) Eigenvalues of the Laplace operator (λ solving $-\Delta\phi = \lambda\phi$) correspond to frequencies we ~~we~~ observe when a string vibrates as governed by the wave equation with fixed ends.

True - Section 5.2 of the text or lecture 9

E.) If $U \subseteq \mathbb{R}^n$ is a bdd-domain with piecewise-C^1 bdry,

$$\int_U u\Delta u\,dx = \int_{\partial U} \frac{\partial u}{\partial n}\,dS + \int_U |\nabla u|^2\,dx$$

$\boxed{\text{False - } \int_U u\Delta u\,dx = \int_{\partial U} u\,\partial u/\partial n\,dS - \int_U |\nabla u|^2\,dx.}$

P2) Let $u(t,x)$ Solve $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 1$ in the first quadrant $Q_1 = \{(t,x) \mid t \geq 0, x \geq 0\}$. Suppose

$$\begin{cases} u(0,x) = x & x \geq 0 \\ u(t,0) = t & t \geq 0 \end{cases}$$

Find a formula for $u \in C^1(Q_1)$ using the method of characteristics.

Solution: HW 1 P2 & 3

The characteristic ODE is $\dot{x}(t) = 1$ with $x(0) = x_0$ or $x(t_0) = 0$ depending on boundary data.

Then, $x(t) = x_0 + t$ for $x_0 \geq 0$

or $x(t) = t - t_0$ for $t_0 \geq 0$

Next, $\frac{Du}{Dt} = 1$ gives $u(t, x(t)) = \begin{cases} t + u(0, x_0) & x \geq t \\ t - t_0 + u(t_0, 0) & x \leq t \end{cases}$

So $u(t,x) = \begin{cases} t + x - t & x \geq t \\ t - t_0 + t_0 & x \leq t \end{cases} = \begin{cases} x & x \geq t \\ t & x \leq t \end{cases}$

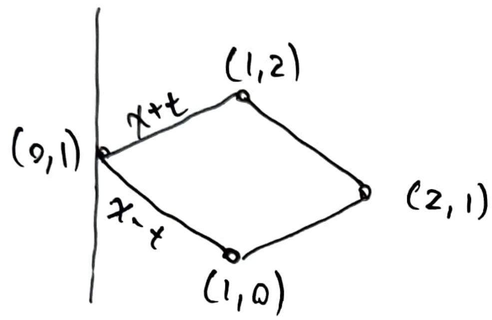
P.3) Assume that $u(t,x) \in C^2([0,\infty)_t \times \mathbb{R}_x)$ solves

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

A.) Let $u(0,1) = 2$, $u(1,2) = 4$, and $u(1,0) = 3$.

What is $u(2,1)$.

Solution: As in HW 2 P1, these points form a parallelogram whose sides are characteristics



By the formula from that problem, $u(2,1) = u(1,2) + u(1,0) - u(0,1)$
$= 7 - 2 = 5.$

B.) Let V solve $\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = 1$ and $V(0,x) = u(0,x)$,

$\partial_t V(0,x) = \partial_t u(0,x)$. What is $V(2,1)$?

Solution:

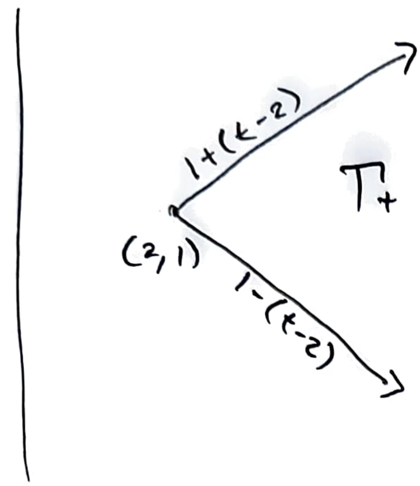
Since $V - u$ solves
$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 1 \\ w(0,x) = 0 \\ \partial_t w(0,x) = 0 \end{cases},$$

$$V(t,x) = u(t,x) + w(t,x) = u(t,x) + \frac{1}{2}\int_{D_{t,x}} 1 \, ds\, d\tau.$$

Then, $V(2,1) = 5 + \frac{1}{2} \cdot \text{Area}\big(\cancel{\{(t,x) : 1-(2-t) \le}$
$\{(t,x) : 1-(2-t) \le x \le 1+(2-t)\}\big)$

$= 5 + \frac{1}{2}(4) = 7.$

C.) What is the range of influence of (2,1)?



$T^+_{(2,1)} = \{(t,x) \mid 3-t \le x \le t-1\}$